

Commutative Algebra of Generalised Frobenius Numbers

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Abstract

We study commutative algebra arising from generalised Frobenius numbers. The k -th (generalised) Frobenius number of natural numbers (a_1, \dots, a_n) is the largest natural number that cannot be written as a non-negative integral combination of (a_1, \dots, a_n) in k distinct ways. Suppose that L is the lattice of integer points of $(a_1, \dots, a_n)^\perp$. Taking cue from the concept of lattice modules due to Bayer and Sturmfels, we define generalised lattice modules $M_L^{(k)}$ whose Castelnuovo-Mumford regularity captures the k -th Frobenius number of (a_1, \dots, a_n) . We study the sequence $\{M_L^{(k)}\}_{k=1}^\infty$ of generalised lattice modules providing an explicit characterisation of their minimal generators. We show that there are only finitely many isomorphism classes of generalised lattice modules. As a consequence of our commutative algebraic approach, we show that the sequence of generalised Frobenius numbers forms a generalised arithmetic progression. We also construct an algorithm to compute the k -th Frobenius number.

1 Introduction

The Frobenius number $F(a_1, \dots, a_n)$ of a collection (a_1, \dots, a_n) of natural numbers such that $\gcd(a_1, \dots, a_n) = 1$ is the largest natural number that cannot be expressed as a non-negative integral linear combination of a_1, \dots, a_n . Note that the condition $\gcd(a_1, \dots, a_n) = 1$ ensures that a sufficiently large integer can be written as a non-negative integral combination of a_1, \dots, a_n . The Frobenius number has been studied extensively from several viewpoints including discrete geometry [12], analytic number theory [6] and commutative algebra [15].

The Frobenius number can be rephrased in the language of lattices as follows [17]. We start by letting $L(a_1, \dots, a_n)$ be a sublattice of the dual lattice $(\mathbb{Z}^n)^*$ of points that evaluate to zero at $(a_1, \dots, a_n) \in \mathbb{Z}^n$. The Frobenius number is precisely the largest integer r such that there exists a point $\mathbf{p} \in (\mathbb{Z}^n)^*$ that evaluates to r at (a_1, \dots, a_n) and \mathbf{p} does not dominate any point in $L(a_1, \dots, a_n)$. Here the domination is according to the partial order induced by the standard basis on $(\mathbb{Z}^n)^*$.

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This leads to a commutative algebraic interpretation of the Frobenius number that we now recall. Let \mathbb{K} be an arbitrary field and let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n -variables with coefficients in \mathbb{K} . Let $I_{L(a_1, \dots, a_n)}$ or simply I_L be the lattice ideal associated to L . Recall that for a sublattice L of \mathbb{Z}^n , the lattice ideal I_L is the ideal generated by all binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that $\mathbf{u} - \mathbf{v} \in L$ and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_{\geq 0}^n$. Note that $L(a_1, \dots, a_n)$ is, by construction, a sublattice of $(\mathbb{Z}^n)^*$. We use the standard isomorphism between $(\mathbb{Z}^n)^*$ and \mathbb{Z}^n to regard it as a sublattice of \mathbb{Z}^n and associate a lattice ideal to it. Observe that S/I_L is naturally graded by the semigroup $Q = \mathbb{N}\{a_1, \dots, a_n\}$.

Theorem 1.1. [15] *The Frobenius number $F(a_1, \dots, a_n)$ is the given by the formula:*

$$\operatorname{reg}_Q(S/I_L) + n - 1 - \sum_{i=1}^n a_i$$

where $\operatorname{reg}_Q(S/I_L)$ is the Castelnuovo-Mumford regularity of S/I_L with respect to its grading by Q . In other words, the Frobenius number is the maximum Q -degree of the highest Betti number of $I_{L(a_1, \dots, a_n)}$ as an S -module subtracted by $\sum_i a_i$.

Remark 1.2. For a Q -graded module M , the invariant $\operatorname{reg}_Q(S/I_L) + n - 1 - \sum_{i=1}^n a_i$ is also called the a -invariant of the module, [15]. \square

Example 1.3. Consider the lattice $L = (3, 5, 8)^\perp \cap \mathbb{Z}^3$. We calculate its corresponding lattice ideal $I_L = \langle x_3 - x_1x_2, x_2^3 - x_1^5 \rangle$. The Betti table corresponding to the minimal free resolution of S/I_L has 22 rows and 3 columns, hence $\operatorname{reg}_Q(S/I_L) = 21$ and $F(3, 5, 8) = 21 + 2 - 16 = 7$. \square

Through this paper by the Castelnuovo-Mumford regularity of a graded module M , we mean the maximum row index in its graded Betti table minus one. If $c_{i,j}$ is the twist corresponding to the Betti number $\beta_{i,j}$, the regularity is given by $\max_{i,j} \{c_{i,j} - i\}$. We refer to Eisenbud [10, Chapter 4] for more information on this topic.

Theorem 1.1 motivates studying “explicit” free resolutions of I_L as an S -module. By an explicit free resolution, we mean a cell complex on L whose relabeling gives a free resolution. For instance, the hull complex [4] gives an explicit (non-minimal, in general) free resolution. We refer to the first section of Miller and Sturmfels [13] for more information.

1.1 Generalised Frobenius Numbers

Recently, the following generalisation of the Frobenius number called the k -th Frobenius number has been proposed [7]. For a natural number k , the k -th Frobenius number $F_k(a_1, \dots, a_n)$ of a collection (a_1, \dots, a_n) of natural numbers such that $\gcd(a_1, \dots, a_n) = 1$ is the largest natural number that cannot be written as k distinct non-negative integral linear combinations of a_1, \dots, a_n . Hence, the first Frobenius number $F_1(a_1, \dots, a_n)$ is the Frobenius number of (a_1, \dots, a_n) . The finiteness of $F_k(a_1, \dots, a_n)$ for all natural numbers k follows by an argument similar to the one for $F_1(a_1, \dots, a_n)$. In the language of lattices, the k -th Frobenius number is the largest integer r such that there exists a point $\mathbf{p} \in (\mathbb{Z}^n)^*$ that evaluates to r

at (a_1, \dots, a_n) and \mathbf{p} does not dominate k distinct points in $L(a_1, \dots, a_n)$. As in the case $k = 1$, the domination is according to the partial order induced by the standard basis on $(\mathbb{Z}^n)^*$.

This interpretation allows a generalisation to any finite index sublattice H of $L(a_1, \dots, a_n)$. The k -th Frobenius number of H is the largest integer r such that there exists a point $\mathbf{p} \in (\mathbb{Z}^n)^*$ that evaluates to r at (a_1, \dots, a_n) and \mathbf{p} does not dominate k distinct points in H . The finite index assumption is necessary for the k -th Frobenius number to be finite. All our results hold in this level of generality.

Our goal in this paper is to develop commutative algebra arising from the k -th Frobenius number. A guiding problem for us is the classification of sequences of generalised Frobenius numbers:

Problem 1.4. (Classification of Frobenius Number Sequences) Given a sequence of natural numbers $\{c_n\}_{n=1}^\infty$, does there exist a vector $(a_1, \dots, a_n) \in \mathbb{N}^n$ and a finite index sublattice H of $L(a_1, \dots, a_n)$ whose sequence of generalised Frobenius numbers is equal to $\{c_n\}_{n=1}^\infty$?

To the best of our knowledge, this problem is wide open. For instance, previous to this it was not known whether a geometric progression with common ratio strictly greater than one can occur as a sequence of Frobenius numbers. As a corollary to our results, we show that the answer to this question is “no”.

We start by recalling another commutative algebraic interpretation of the Frobenius number $F_1(a_1, \dots, a_n)$ following Bayer and Sturmfels [4], [13]. The key concepts here are the group algebra $S[L]$ and the lattice module M_L associated to L .

The group algebra $S[L]$ is the \mathbb{K} -algebra generated by Laurent monomials $\mathbf{x}^{\mathbf{u}} \cdot \mathbf{z}^{\mathbf{v}}$ such that $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{N}^n$, $\mathbf{v} = (v_1, \dots, v_n) \in L$ where $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{z}^{\mathbf{v}}$ are Laurent monomials $x_1^{u_1} \dots x_n^{u_n}$ and $z_1^{v_1} \dots z_n^{v_n}$. The lattice module M_L is the S -module generated by Laurent monomials $\mathbf{x}^{\mathbf{w}}$ over all $\mathbf{w} \in L$. We can realise M_L as a cyclic $S[L]$ -module as follows:

$$M_L \cong S[L] / \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \cdot \mathbf{z}^{\mathbf{u}-\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{N}^n, \mathbf{u} - \mathbf{v} \in L \rangle$$

To see this isomorphism, consider the morphism from $S[L]$ to M_L that takes $\mathbf{x}^{\mathbf{u}} \cdot \mathbf{z}^{\mathbf{v}}$ to $\mathbf{x}^{\mathbf{u}+\mathbf{v}}$ and use the first isomorphism theorem.

The group algebra $S[L]$ is naturally \mathbb{Z}^n -graded where the graded piece indexed by $\mathbf{u} \in \mathbb{Z}^n$ is the \mathbb{K} -vector space spanned by $\{\mathbf{x}^{\mathbf{u}} \cdot \mathbf{z}^{\mathbf{v}}\}_{\mathbf{v} \in L}$. The lattice module M_L is also naturally \mathbb{Z}^n -graded, since it is generated by Laurent monomials. For the lattice $L(a_1, \dots, a_n) \subset (\mathbb{Z}^n)^*$, we again regard it as a sublattice of \mathbb{Z}^n via the standard isomorphism between $(\mathbb{Z}^n)^*$ and \mathbb{Z}^n to associate the group algebra and the lattice module to it. In this case, the \mathbb{Z}^n -grading on M_L also yields a \mathbb{Z} -grading, corresponding to the evaluation $\sigma \rightarrow \sigma((a_1, \dots, a_n))$ for $\sigma \in (\mathbb{Z}^n)^*$. We refer to this grading as the (a_1, \dots, a_n) -weighted grading on M_L or simply the weighted grading on M_L . Note that we also have both these gradings for any finite index sublattice of $L(a_1, \dots, a_n)$.

Note that S is a $S[L]$ -module via the isomorphism $S \cong S[L] / \langle \mathbf{z}^{\mathbf{u}} - 1_{\mathbb{K}} \mid \mathbf{u} \in L \rangle$. Bayer and Sturmfels show that there is a categorical equivalence between \mathbb{Z}^n -graded $S[L]$ -modules to \mathbb{Z}^n/L -graded S -modules. The functor π realising this equivalence tensors an $S[L]$ -module

with S (here S is seen as an $S[L]$ -module). The functor π takes M_L to S/I_L . Hence, a \mathbb{Z}^n/L -minimal free resolution of S/I_L as an S -module can be obtained by applying the functor π to a \mathbb{Z}^n -graded minimal free resolution of M_L as an $S[L]$ -module, we refer to Miller and Sturmfels [13] for an example. Hence, we have the following interpretation of the Frobenius number in terms of the lattice module M_L .

Theorem 1.5. [4], [15] *The Frobenius number $F(a_1, \dots, a_n)$ is*

$$\operatorname{reg}_{\mathbb{Z}^n}(M_L) + n - 1 - \sum_{i=1}^n a_i$$

where $\operatorname{reg}_{\mathbb{Z}^n}(M_L)$ is the Castelnuovo-Mumford regularity of M_L with respect to its \mathbb{Z}^n -grading.

The module M_L behaves similar to a monomial ideal. This categorical equivalence can be used to transfer homological constructions from M_L to S/I_L . By applying the functor π to M_L and noting that the \mathbb{Z}^n/L -grading coincides with the Q -grading on L , we obtain Theorem 1.1. We start by generalising Theorem 1.5 to k -th Frobenius numbers. We generalise the lattice module M_L to the k -th lattice module $M_L^{(k)}$ as follows.

The k -th lattice module $M_L^{(k)}$ is the S -module generated by Laurent monomials $\mathbf{x}^{\mathbf{w}}$ such that \mathbf{w} dominates at least k lattice points.

By construction, the first lattice module $M_L^{(1)}$ is the lattice module M_L . The module $M_L^{(k)}$ is a finitely generated $S[L]$ -module with the $S[L]$ -action given by $\mathbf{x}^{\mathbf{u}}\mathbf{z}^{\mathbf{v}} \cdot \mathbf{x}^{\mathbf{w}} = \mathbf{x}^{\mathbf{w}+\mathbf{u}+\mathbf{v}}$ where $\mathbf{x}^{\mathbf{u}}\mathbf{z}^{\mathbf{v}} \in S[L]$ and $\mathbf{x}^{\mathbf{w}} \in M_L^{(k)}$. The generalised lattice module $M_L^{(k)}$ also carries a \mathbb{Z}^n -grading since it is generated by Laurent monomials. However, in general the module $M_L^{(k)}$ is not a cyclic module for natural numbers $k > 1$. We have the following commutative algebraic characterisation of generalised Frobenius numbers in terms of the generalised lattice modules.

Proposition 1.6. *The k -th Frobenius number of (a_1, \dots, a_n) is given by the formula:*

$$\operatorname{reg}_{\mathbb{Z}^n}(M_L^{(k)}) + n - 1 - \sum_{i=1}^n a_i$$

where $L := L(a_1, \dots, a_n)$ and $\operatorname{reg}_{\mathbb{Z}^n}(M_L^{(k)})$ is the Castelnuovo-Mumford regularity of the $S[L]$ -module $M_L^{(k)}$ with respect to its \mathbb{Z}^n -grading.

Proposition 1.6 follows from two observations. Let $c_{i,j}$ be the twist of the free module corresponding to the graded Betti number $\beta_{i,j}$ of $\pi(M_L^{(k)})$. A computation similar to the proof of [15, Theorem 3.1], comparing expressions for the Hilbert series of $\pi(M_L^{(k)})$ gives the following expression for the k -th Frobenius number:

$$F_k = \max_{i,j} c_{i,j} - \sum_i a_i$$

The second observation is that $\pi(M_L^{(k)})$ is a module Cohen-Macaulay with both Krull dimension and depth equal to one. This implies that the regularity and $\max_{i,j} c_{i,j}$ are attained at the highest homological degree, in this case $n - 1$, Proposition 1.6 follows as an immediate consequence.

While Proposition 1.6 provides a simple description of the generalised Frobenius number in terms of $M_L^{(k)}$, it has a number of limitations. For instance, given a natural number k Proposition 1.6 is not directly useful to determine the k -th Frobenius number since it relies on the explicit knowledge of $M_L^{(k)}$. The generalised lattice modules are naturally related by the filtration:

$$M_L^{(1)} \supseteq M_L^{(2)} \supseteq M_L^{(3)} \dots$$

However, Proposition 1.6 does not capture this. The connection between the generalised Frobenius number can be better understood by studying the interlink between the generalised lattice modules. With this in mind, we delve into a detailed study of the generalised lattice modules: their minimal generating sets, their Hilbert series and their syzygies. We now provide a brief description of our results.

We associate a graph G_L on L as follows. Fix a binomial minimal generating set of I_L . The graph G_L is defined as follows

There is an edge between points \mathbf{w}_1 and \mathbf{w}_2 in L if there exists a binomial minimal generator $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that the difference of its exponents is equal to $\mathbf{w}_1 - \mathbf{w}_2$ i.e.,

$$\mathbf{u} - \mathbf{v} = \mathbf{w}_1 - \mathbf{w}_2 .$$

By construction, G_L has an L -action on its edges since if $(\mathbf{w}_1, \mathbf{w}_2)$ is an edge then $(\mathbf{w}_1 + \mathbf{y}, \mathbf{w}_2 + \mathbf{y})$ for any $\mathbf{y} \in L$.

Let d_{G_L} be the metric on L induced by the graph G_L . For a point $\mathbf{w} \in L$, let $N^{(k)}(\mathbf{w})$ be the set of all points in L in the ball of radius k centered at \mathbf{w} in the metric d_{G_L} .

Theorem 1.7. (Neighbourhood Theorem) *For any non-negative integer k , any minimal generator of $M_L^{(k+1)}$ as an $S[L]$ -module is the least common multiple of Laurent monomials corresponding to $(k+1)$ lattice points each of which is a point in $N^{(k)}(\mathbf{0})$ where $\mathbf{0} = (0, \dots, 0)$.*

We prove Theorem 1.7 by an inductive characterisation of the generalised lattice modules. This characterisation of $M_L^{(k+1)}$ is in terms of syzygies of $M_L^{(k)}$ that we believe is of independent interest. We briefly describe this characterisation in the following. Fix a natural number k , a minimal generator $\mathbf{x}^{\mathbf{w}}$ of $M_L^{(k)}$ as an S -module is called exceptional if \mathbf{w} dominates strictly larger than k points in L . We describe $M_L^{(k+1)}$ in terms of the exceptional generators of $M_L^{(k)}$ and the first syzygies of a “modification” of $M_L^{(k)}$ that we now describe. Let $M_{L,\text{mod}}^{(k)}$ be the S -module generated by every element of $M_L^{(k)}$ and the element $1_{\mathbb{K}}$ (the multiplicative identity of \mathbb{K}). Formally,

$$M_{L,\text{mod}}^{(k)} = \langle 1_{\mathbb{K}}, m \mid m \in M_L^{(k)} \rangle_S$$

Note that $M_{L,\text{mod}}^{(k)}$ is naturally an S -module but not an $S[L]$ -module i.e., $M_{L,\text{mod}}^{(k)}$ does not inherit the natural L -action and hence, is not an $S[L]$ -module.

By construction, we have the following characterisation of minimal generators of $M_{L,\text{mod}}^{(k)}$:

Proposition 1.8. *The (Laurent) monomial minimal generating set of $M_{L,\text{mod}}^{(k)}$ consists of precisely $1_{\mathbb{K}}$ and every (Laurent) monomial minimal generator of $M_L^{(k)}$ that is not divisible by $1_{\mathbb{K}}$ (in other words, whose exponent does not dominate the origin).*

For each minimal generator g_1 of $M_{L,\text{mod}}^{(k)}$, let $\text{Syz}_{g_1}^1(M_{L,\text{mod}}^{(k)})$ be the \mathbb{K} -vector space $\text{Syz}_{g_1}^1(M_{L,\text{mod}}^{(k)})$ generated by syzygies of the form:

$$m \cdot (0, \dots, 0, \underbrace{\text{lcm}(g_1, g_2)/g_1}_{g_1}, 0, \dots, 0, \underbrace{-\text{lcm}(g_1, g_2)/g_2}_{g_2}, 0, \dots, 0)$$

where $g_2 \neq g_1$ is a minimal generator of $M_{L,\text{mod}}^{(k)}$ and m is a monomial in S . Note that multiplication by m is the standard multiplication on S . Consider the direct sum $\oplus_g \text{Syz}_g^1(M_{L,\text{mod}}^{(k)})$ where g varies over all minimal generators of $M_{L,\text{mod}}^{(k)}$.

We define a map $\phi_S^{(k)}$ from $\oplus_g \text{Syz}_g^1(M_{L,\text{mod}}^{(k)})$ to $M_L^{(k+1)}$. We first define the map $\phi_S^{(k)}$ from the canonical basis of each piece $\text{Syz}_g^1(M_{L,\text{mod}}^{(k)})$ to $M_L^{(k+1)}$ as follows:

$$\phi_S^{(k)}((s_1, s_2)) = \mathbf{x}^{\deg_{\mathbb{Z}^n}((s_1, s_2))} \text{ where } \deg_{\mathbb{Z}^n}(\cdot) \text{ is the multidegree.}$$

We extend this map \mathbb{K} -linearly to define $\phi_S^{(k)}$. Note that the image of $\phi_S^{(k)}$ is an element of $M_L^{(k+1)}$. This is because (s_1, s_2) is of the form $(\text{lcm}(g_1, g_2)/g_2, -\text{lcm}(g_1, g_2)/g_1)$ for two distinct minimal generators of $M_{L,\text{mod}}^{(k)}$. By construction, $\phi_S^{(k)}((s_1, s_2)) = \text{lcm}(g_1, g_2)$. By Proposition 2.6, we have the following two cases: either both g_1 and g_2 are minimal generators of $M_L^{(k)}$ or one of them $g_1 = 1_{\mathbb{K}}$, say and g_2 is a minimal generator of $M_L^{(k)}$ that is not divisible by $1_{\mathbb{K}}$. In both cases, the support of $\text{lcm}(g_1, g_2)$ contains at least $(k+1)$ points in L (by support of Laurent monomial, we mean the set of points in L that its exponent dominates). It contains (potentially among others) the unions of the supports of g_1 and g_2 . Hence, the image of $\phi_S^{(k)}$ is in $M_L^{(k+1)}$. We show the following converse to this.

Theorem 1.9. *Up to the action of L , every minimal generator of $M_L^{(k+1)}$ is either in the image of $\phi_S^{(k)}$ or is an exceptional generator of $M_L^{(k)}$.*

Example 1.10. Consider the lattice $(3, 5, 8)^\perp \cap \mathbb{Z}^3$ with corresponding lattice ideal $I_L = \langle x_3 - x_1x_2, x_2^3 - x_1^5 \rangle$. As $1_{\mathbb{K}}$ is a minimal generator of $M_L^{(1)}$, the lattice module is not altered under the modification construction. The minimal first syzygies of $M_L^{(1)}$, up to the action of L , are of the form $(\text{lcm}(\mathbf{x}^{\mathbf{u}}, 1_{\mathbb{K}})/1_{\mathbb{K}}, -\text{lcm}(\mathbf{x}^{\mathbf{u}}, 1_{\mathbb{K}})/\mathbf{x}^{\mathbf{u}})$ where $\mathbf{u} \in N^{(1)}(\mathbf{0})$. The map $\phi_S^{(1)}$ sends the minimal first syzygies to $\{x_3, x_1x_2, x_2^3, x_1^5\}$, precisely the monomials in each minimal binomial of I_L . This gives us an explicit description of $M_L^{(2)} = \langle x_3, x_2^3 \rangle$ as an $S[L]$ -module and a minimal generating set. \square

Theorem 1.9 characterises the minimal generators of $M_L^{(k)}$, a natural next question is about the syzygies of $M_L^{(k)}$ as an $S[L]$ -module. Is there a similar inductive characterisation of the syzygies of $M_L^{(k)}$? What are the possible Betti tables of $M_L^{(k)}$? Both these questions are wide open in general. As a first result in this direction, we show the following finiteness result. Recall that for a \mathbb{Z}^n -graded module M (either an S -module or an $S[L]$ -module) and for any $\mathbf{b} \in \mathbb{Z}^n$, we have the twist $M(-\mathbf{b})$ of M defined by $(M_{-\mathbf{b}})_{\mathbf{c}} = M_{\mathbf{b}+\mathbf{c}}$ for every $\mathbf{c} \in \mathbb{Z}^n$.

Theorem 1.11. *Let L be a finite index sublattice of $(a_1, \dots, a_n)^\perp \cap \mathbb{Z}^n$. For each $k \in \mathbb{N}$, let $\mathbf{x}^{\mathbf{u}_k}$ be any element of $M_L^{(k)}$ of the smallest (a_1, \dots, a_n) -weighted degree. There are finitely many classes among the generalised lattice modules $\{M_L^{(k)}(-\mathbf{u}_k)\}_{k \in \mathbb{N}}$ up to isomorphism of both \mathbb{Z}^n -graded $S[L]$ -modules and \mathbb{Z}^n -graded S -modules. Hence, there are only finitely many distinct Betti tables for the generalised lattice modules of L .*

The key object in the proof of Theorem 1.11 is a poset that we refer to as the structure poset associated to L . The elements of the structure poset of L are elements in \mathbb{Z}^n/L of (a_1, \dots, a_n) -weighted degree in the range $[0, F_1]$ where F_1 is the first Frobenius number of L . Note that there are precisely $\text{ind}(L) \cdot (F_1 + 1)$ elements in the structure poset, where $\text{ind}(L)$ is the index of L in $(a_1, \dots, a_n)^\perp \cap \mathbb{Z}^n$. The partial order in this poset is defined as follows: for elements $[\mathbf{a}]$, $[\mathbf{b}]$ in the structure poset we say that $[\mathbf{a}] \geq [\mathbf{b}]$ if for every representative $\mathbf{a} \in \mathbb{Z}^n$ of $[\mathbf{a}]$ there exists a representative \mathbf{b} of $[\mathbf{b}]$ such that $\mathbf{a} \geq \mathbf{b}$.

Let m_k be the minimum weighted degree of any element of $M_L^{(k)}$. The key observation in the proof of Theorem 1.11 is that $M_L^{(k)}$ is completely determined (up to isomorphism of \mathbb{Z}^n -graded $S[L]$ -modules) by “filling” the structure poset of L . More precisely, to determine $M_L^{(k)}$ we need to know the elements in \mathbb{Z}^n/L of weighted degree $[m_k, m_k + F_1]$ that dominate at least k points in L . This is given by the values of the Hilbert function of the polynomial ring with the \mathbb{Z}^n/L -grading. These elements determine a subposet of the structure poset of L that we refer to as the *structure poset of $M_L^{(k)}$* . The structure poset of $M_L^{(k)}$ determines it up to isomorphism of \mathbb{Z}^n -graded $S[L]$ -modules. Since the structure poset of L is finite it has only finitely many subposets. Hence, there are only finitely many \mathbb{Z}^n -graded isomorphism classes of generalised lattice modules. A classification of subposets of the structure poset of L that can occur as structure posets of generalised lattice modules is wide open. In Section 3, we provide a detailed example illustrating this phenomenon. As a corollary to Theorem 1.11 we obtain the following:

Corollary 1.12. *There exists a finite set of integers $\{b_1, \dots, b_t\} \subset \mathbb{Z}_{\geq 0} \cup \{-1\}$ such that for every k there exists a natural number j such that the k -th Frobenius number can be written as:*

$$F_k = m_k + b_j$$

where m_k is the minimum (a_1, \dots, a_n) -weighted degree of any element in $M_L^{(k)}$. This finite set $\{b_1, \dots, b_t\}$ is the precisely the set of natural numbers that can be realised as $\text{reg}(M_L^{(k)}(-\mathbf{u}_k)) - n - 1 + \sum_{i=1}^n a_i$.

For $k = 1$, note that $m_k = 0$ and $b_j = \text{reg}_{\mathbb{Z}^n}(M_L^{(1)}) + n - 1 - \sum a_i$. Suppose that L is a finite sublattice of $(a_1, a_2)^\perp \cap \mathbb{Z}^2$ where (a_1, a_2) are relatively prime numbers. The set $\{b_1, \dots, b_t\}$ consists of one element and the generalised lattice module $M_L^{(k)}$ is generated by one element. When $L = (a_1, a_2)^\perp \cap \mathbb{Z}^2$, the k -th lattice module $M_L^{(k)}$ is generated by $x_1^{a_2} x_2^{(k-2)a_1}$. Hence, $m_k = (k-1)a_1 a_2$ and the set $\{b_1, \dots, b_t\}$ consists of only one element $F_1 = a_1 a_2 - a_1 - a_2$. The k -th Frobenius number F_k will be $m_k + F_1 = k a_1 a_2 - a_1 - a_2$, exactly as in [7]. Obtaining a formula along the lines of Corollary 1.12 was suggested as an open problem in [7]. Finally, note that if the structure poset of $M_L^{(k)}$ is equal to the structure poset of L then $F_k = m_k - 1$ and in this case $b_k = -1$.

As an application of Theorem 1.11, we show that the sequence of Frobenius numbers $(F_k)_{k=1}^\infty$ is a generalised arithmetic progression. Recall that a sequence $(c_k)_{k=1}^\infty$ is called a generalised arithmetic progression if there exists a finite set such that for every $k \in \mathbb{N}$ the difference $c_{k+1} - c_k$ is contained in this set. This provides a partial answer to Problem 1.4

Theorem 1.13. *For any finite index sublattice L of $(a_1, \dots, a_n)^\perp \cap \mathbb{Z}^n$, the sequence of Frobenius numbers $(F_k)_{k=1}^\infty$ is a generalised arithmetic progression.*

We prove Theorem 1.13 using Corollary 1.12 along with the fact that the sequence $(m_k)_{k=1}^\infty$ is a generalised arithmetic progression. As a corollary, a geometric progression with common ratio strictly greater than one cannot occur as a sequence of Frobenius numbers.

As another application of our results, we use the neighbourhood theorem (Theorem 1.7) to construct an algorithm that takes the lattice in terms of a basis and a natural number k as input, computes the k -th lattice module and the k -th Frobenius number.

1.2 Related Work

There is a vast literature on the Frobenius number, we refer to Alfonsín's book [16] for more information. Work on the generalised Frobenius numbers has so far primarily used analytic methods and methods from polyhedral geometry. The work of Beck and Robins [7] uses analytic methods to derive an explicit formula for the coefficients of the Hilbert series of $\mathbb{K}[x, y]$ with (a_1, a_2) -weighted grading. Aliev, Fukshansky and Henk [20] give bounds generalising a theorem of Kannan for the first Frobenius number. They relate the k -th Frobenius number to the k -covering radius of a simplex with respect to the lattice $(a_1, \dots, a_n)^\perp \cap \mathbb{Z}^n$. Hence, they achieved bounds on the generalised Frobenius number using methods from the geometry of numbers.

A recent work of Aliev, De Loera and Louveaux [3] considered the semigroup

$$\text{Sg}_{\geq k}((a_1, \dots, a_n)) = \{b : \text{there exist distinct } \mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{N}^n \text{ such that } (a_1, \dots, a_n) \cdot \mathbf{x}_i = b\}$$

In this framework, the k -th Frobenius number is the largest non-negative integer $b \notin \text{Sg}_{\geq k}((a_1, \dots, a_n))$. They study this semigroup by considering the monomial ideal $I^{(k)}((a_1, \dots, a_n))$ such that the set of (a_1, \dots, a_n) -weighted degrees of its elements is equal to $\text{Sg}_{\geq k}((a_1, \dots, a_n))$ [3, Theorem 1]. They use the Gordon-Dickson Lemma to deduce the

finite generation of $I^{(k)}((a_1, \dots, a_n))$ and hence, $\text{Sg}_{\geq k}((a_1, \dots, a_n))$. In fact, [3] study a more general version where (a_1, \dots, a_n) is replaced by any $d \times n$ matrix with integer entries.

The monomial ideal $I^{(k)}((a_1, \dots, a_n))$ is, in fact, the intersection of $M_L^{(k)}$ with the polynomial ring S . We note that this ideal $I^{(k)}((a_1, \dots, a_n))$ does not carry an L -action and this seems to make it less amenable to study compared to $M_L^{(k)}$.

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2 Generalised Lattice Modules

In this section, we discuss generalised lattice modules in detail including the neighbourhood theorem (Theorem 1.7) and the inductive characterisation of $M_L^{(k)}$ (Theorem 1.9) in the introduction. We start by recalling the definition of generalised lattice modules. Fix a non-zero vector $(a_1, \dots, a_n) \in \mathbb{N}^n$. Let L be a finite index sublattice of the lattice of integer points in $(a_1, \dots, a_n)^\perp \cap \mathbb{Z}^n$ and $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n -variables.

Definition 2.1. *The k -th lattice module $M_L^{(k)}$ is the S -module generated by Laurent monomials $\mathbf{x}^{\mathbf{w}}$ where \mathbf{w} is an element in \mathbb{Z}^n that dominates at least k points in L . Formally,*

$$M_L^{(k)} = \langle \mathbf{x}^{\mathbf{w}} \mid \mathbf{w} \in \mathbb{Z}^n \text{ dominates at least } k \text{ points in } L \rangle_S$$

By construction, $M_L^{(k)}$ is a \mathbb{Z}^n -graded S -module and is \mathbb{Z} -graded by (a_1, \dots, a_n) -weighted degree. On the other hand, $M_L^{(k)}$ is not a finitely generated S -module. However $M_L^{(k)}$ carries an L -action via the map $\mathbf{v} \cdot \mathbf{x}^{\mathbf{w}} = \mathbf{x}^{\mathbf{w}+\mathbf{v}}$ for every $\mathbf{v} \in L$ and $\mathbf{x}^{\mathbf{w}} \in M_L^{(k)}$. This action makes $M_L^{(k)}$ into a L -module and furthermore, into an $S[L]$ -module where $S[L]$ is the group algebra of L .

Recall that the group algebra $S[L]$ is defined as the \mathbb{K} -algebra generated by symbols $\mathbf{x}^{\mathbf{u}}\mathbf{z}^{\mathbf{v}}$ where $\mathbf{u} \in \mathbb{N}^n$ and $\mathbf{v} \in L$ with multiplication given by $\mathbf{x}^{\mathbf{u}_1}\mathbf{z}^{\mathbf{v}_1} \cdot \mathbf{x}^{\mathbf{u}_2}\mathbf{z}^{\mathbf{v}_2} = \mathbf{x}^{\mathbf{u}_1+\mathbf{u}_2}\mathbf{z}^{\mathbf{v}_1+\mathbf{v}_2}$. The action by $S[L]$ on the k -th lattice module $M_L^{(k)}$ is given by $\mathbf{x}^{\mathbf{u}}\mathbf{z}^{\mathbf{v}} \cdot \mathbf{x}^{\mathbf{w}} = \mathbf{x}^{\mathbf{u}+\mathbf{v}+\mathbf{w}}$ where $\mathbf{x}^{\mathbf{u}}\mathbf{z}^{\mathbf{v}} \in S[L]$ and $\mathbf{x}^{\mathbf{w}} \in M_L^{(k)}$. We refer to [4], [13] for a more detailed discussion on this topic. In the following, we show that $M_L^{(k)}$ is a finitely generated $S[L]$ -module.

Proposition 2.2. *For any natural number k , the k -th lattice module $M_L^{(k)}$ is a finite generated $S[L]$ -module.*

Proof. By the action of $S[L]$ on $M_L^{(k)}$, it suffices to consider orbit representatives of the L -action on $M_L^{(k)}$ that dominate the origin. These representatives are monomials in S (rather than Laurent monomials) and define a monomial ideal in the polynomial ring S . By the Gordon-Dickson lemma, this monomial ideal is finitely generated and hence $M_L^{(k)}$ is finitely generated as an $S[L]$ -module. \square

The above proof is based on an argument in [3], it is however not constructive in the sense that it does not give bounds on the degrees of the minimal generators of $M_L^{(k)}$. The methods in Section 3 give a constructive proof that shows that the (a_1, \dots, a_n) -weighted degree of any minimal generator of $M_L^{(k)}$ is in the interval $[m_k, m_k + F_1]$ where m_k is the minimum degree of a Laurent monomial $\mathbf{x}^{\mathbf{w}}$ such that \mathbf{w} dominates at least k lattice points.

Example 2.3. The case $k = 1$ is precisely the notion of lattice module studied by Bayer and Sturmfels [4]. The lattice module $M_L^{(1)}$ as an S -module is generated by Laurent monomials $\mathbf{x}^{\mathbf{w}}$ where $\mathbf{w} \in L$. As an $S[L]$ -module, $M_L^{(1)}$ is cyclic and is generated by the element $1_{\mathbb{K}}$. Furthermore, [4] show that $M_L^{(1)} \cong S[L]/\langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mathbf{z}^{\mathbf{u}-\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{N}, \mathbf{u} - \mathbf{v} \in L \rangle$. \square

For $k \geq 2$, the lattice modules $M_L^{(k)}$ are in general not cyclic $S[L]$ -modules. For $k = 2$, we give a simple description of a minimal generating set of $M_L^{(2)}$ in terms of the first syzygies of $M_L^{(1)}$ (Theorem 2.5). For $k \geq 3$, a generalisation of this result is more involved and is the content of Theorem 2.7. One source of complication is that for $k \geq 2$, the lattice modules $M_L^{(k)}$ have exceptional generators i.e., those that dominate strictly greater than k points in L , whereas $M_L^{(1)}$ does not have exceptional generators. Another complication is for $k \geq 3$, we may get minimal generators of $M_L^{(k)}$ that do not arise as a syzygy between two minimal generators of $M_L^{(k-1)}$, rather as a “syzygy between a minimal generator and a lattice point”. This motivates us to consider the syzygies of $M_{L, \text{mod}}^{(k)}$.

2.1 Inductive Characterisation of $M_L^{(2)}$

We discuss the simplest generalised lattice module $M_L^{(2)}$. We start with a description of the minimal generators of $M_L^{(2)}$. Recall from the introduction that the key to this description is the morphism $\phi_S^{(k)}$ between $\oplus_g \text{Syz}_g^1(M_{L, \text{mod}}^{(k)})$ and $M_L^{(k+1)}$. We now describe the specialisation of this map for $k = 1$. We first note that $M_L^{(1)} = M_{L, \text{mod}}^{(1)}$ and that each piece $\text{Syz}_{\mathbf{x}^{\mathbf{u}}}^1(M_L^{(k)})$ has a basis of the form:

$$m \cdot (0, \dots, 0, \underbrace{\text{lcm}(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}})/\mathbf{x}^{\mathbf{u}}}_{\mathbf{x}^{\mathbf{u}}}, \dots, -\underbrace{\text{lcm}(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}})/\mathbf{x}^{\mathbf{v}}}_{\mathbf{x}^{\mathbf{v}}}, 0, \dots, 0),$$

where $\mathbf{x}^{\mathbf{v}} \neq \mathbf{x}^{\mathbf{u}}$ is a Laurent monomial minimal generator in $M_L^{(1)}$ as an S -module, $\text{lcm}(\cdot, \cdot)$ is the least common multiple and m is a monomial in S . Note that multiplication by m is the standard multiplication on S .

We define a map $\phi_S^{(1)}$ on this basis of $\text{Syz}_{\mathbf{x}^{\mathbf{u}}}^1(M_L^{(1)})$ and extend it \mathbb{K} -linearly. The map $\phi_S^{(1)} : \text{Syz}_{\mathbf{x}^{\mathbf{u}}}^1(M_L^{(1)}) \rightarrow M_L^{(2)}$ takes the element $s = (\text{lcm}(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}})/\mathbf{x}^{\mathbf{u}}, -\text{lcm}(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}})/\mathbf{x}^{\mathbf{v}})$ to $\mathbf{x}^{\deg_{\mathbb{Z}^n}(s)}$ where $\deg_{\mathbb{Z}^n}(s)$ is the \mathbb{Z}^n -graded degree of s . In fact, $\deg_{\mathbb{Z}^n}(s) = \max(\mathbf{u}, \mathbf{v})$ where \max is the coordinate-wise maximum. Furthermore, $\mathbf{x}^{\deg_{\mathbb{Z}^n}(s)} \in M_L^{(2)}$ since the point $\max(\mathbf{u}, \mathbf{v})$ dominates at least two lattice points, namely \mathbf{u} and \mathbf{v} . In the following, we note that the map $\phi_S^{(1)}$ is surjective.

Proposition 2.4. *The map $\phi_S^{(1)}$ is surjective.*

Proof. It suffices to prove that every Laurent monomial in $M_L^{(2)}$ can be realised as the image of an element in $\text{Syz}_{\mathbf{x}^{\mathbf{u}}}^1(M_L^{(1)})$ for some minimal generator $\mathbf{x}^{\mathbf{u}}$ of $M_L^{(1)}$. To see this, consider a Laurent monomial $\mathbf{x}^{\mathbf{w}}$ in $M_L^{(2)}$. By the definition of $M_L^{(2)}$, the point \mathbf{w} dominates at least two points in L . Consider any two points \mathbf{u}_1 and \mathbf{u}_2 in L that \mathbf{w} dominates and consider the Laurent monomial $\text{lcm}(\mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{u}_2})$. This is contained in $M_L^{(2)}$ and is the image of $(\text{lcm}(\mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{u}_2})/\mathbf{x}^{\mathbf{u}_1}, -\text{lcm}(\mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{u}_2})/\mathbf{x}^{\mathbf{u}_2}) \in \text{Syz}_{\mathbf{x}^{\mathbf{u}_1}}^1(M_L^{(1)})$ under $\phi_S^{(1)}$. Hence, by multiplying this syzygy by the monomial $\mathbf{x}^{\mathbf{w}}/\text{lcm}(\mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{u}_2})$ we conclude that $\mathbf{x}^{\mathbf{w}}$ is also in the image of $\phi_S^{(1)}$. \square

Proposition 2.4 is not directly amenable for computational purposes since $M_L^{(2)}$ is not finitely generated as an S -module. However, $M_L^{(2)}$ is finitely generated as an $S[L]$ -module. Note that there is a natural L -action on $\oplus_g \text{Syz}_g^1(M_L^{(1)})$ and a surjective map between the first syzygy module of $M_L^{(1)}$ as an $S[L]$ -module and the piece $\text{Syz}_0^1(M_L^{(1)})$. Composing this with $\phi_S^{(1)}$ gives a surjective map $\phi_{S[L]}^{(1)}$ between the first syzygy module of $M_L^{(1)}$ and $M_L^{(2)}$ as $S[L]$ -modules. To explicitly describe the map $\phi_{S[L]}^{(1)}$, we first note that the first syzygy module of $M_L^{(1)}$ as an $S[L]$ -module has a \mathbb{K} -vector space basis of the form:

$$\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mathbf{z}^{\mathbf{u}-\mathbf{v}}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$ and $\mathbf{u} - \mathbf{v} \in L$. The map $\phi_{S[L]}^{(1)}$ takes $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mathbf{z}^{\mathbf{u}-\mathbf{v}}$ to $\mathbf{x}^{\mathbf{u}} \in M_L^{(2)}$. Since the functor π takes $M_L^{(1)}$ to S/I_L and $\text{Syz}^1(S/I_L) = I_L$, this induces a map from any binomial minimal generating set of I_L to $M_L^{(2)}$, we also refer to this map $\phi_{S[L]}^{(1)}$ by an abuse of notation. We obtain the following.

Theorem 2.5. *The lattice module $M_L^{(2)}$ as an $S[L]$ -module is generated by the image of $\phi_{S[L]}^{(1)}$ on a binomial minimal generating set of the lattice ideal I_L .*

As Example 1.10 shows, the map $\phi_S^{(1)}$ is not injective. In general, take a Koszul syzygy between two minimal generators in $M_L^{(2)}$ and note that it does not lift to a syzygy between the corresponding minimal generators in $\text{Syz}^1(M_L^{(1)})$. This shows that $\phi_S^{(1)}$ has a non-trivial kernel and hence is not injective.

2.2 Inductive Characterisation of $M_L^{(k)}$

We generalise Proposition 2.4 to arbitrary lattice modules to obtain an induction characterisation of $M_L^{(k)}$. Let us briefly recall the relevant objects from the introduction.

The modification $M_{L,\text{mod}}^{(k)}$ of $M_L^{(k)}$ is the S -module generated by every element of $M_L^{(k)}$ and the element $1_{\mathbb{K}}$. Hence,

$$M_{L,\text{mod}}^{(k)} = \langle 1_{\mathbb{K}}, m \mid m \in M_L^{(k)} \rangle_S$$

By the construction of $M_{L,\text{mod}}^{(k)}$, we have the following characterisation of its minimal generators.

Proposition 2.6. *The (Laurent) monomial minimal generating set of $M_{L,\text{mod}}^{(k)}$ consists of precisely $1_{\mathbb{K}}$ and every (Laurent) monomial minimal generator of $M_L^{(k)}$ that is not divisible by $1_{\mathbb{K}}$ (in other words, whose exponent does not dominate the origin).*

For a minimal generator g of $M_{L,\text{mod}}^{(k)}$, the map $\phi_S^{(k)}$ from $\text{Syz}_g^1(M_{L,\text{mod}}^{(k)})$ to $M_L^{(k+1)}$ is defined on the canonical basis of $\text{Syz}_g^1(M_{L,\text{mod}}^{(k)})$ as:

$$\phi_S^{(k)}((s_1, s_2)) = \mathbf{x}^{\deg_{\mathbb{Z}^n}((s_1, s_2))} \text{ where } \deg_{\mathbb{Z}^n}(\cdot) \text{ is the multidegree of the syzygy.}$$

We extend the above map \mathbb{K} -linearly to define $\phi_S^{(k)}$. As noted in the introduction, the image of $\phi_S^{(k)}$ is contained in $M_L^{(k+1)}$. Theorem 2.7 is a converse to this.

Suppose that $\mathbf{x}^{\mathbf{w}} \in M_L^{(k+1)}$ is a minimal generator and let $U = \{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ be the set of points in L that \mathbf{w} dominates. For a subset T of size k , let ℓ_T be the least common multiple of the Laurent monomials associated to points in T .

Theorem 2.7. *Up to the action of L , any minimal generator $\mathbf{x}^{\mathbf{w}}$ of $M_L^{(k+1)}$ is either in the image of $\phi_S^{(k)}$ or is an exceptional generator of $M_L^{(k)}$. Furthermore, we have the following classification of minimal generators of $M_L^{(k+1)}$.*

1. *If ℓ_T is the same for every subset T of U of size k then, $\mathbf{x}^{\mathbf{w}}$ is an exceptional generator of $M_L^{(k)}$.*
2. *If there exist subsets T_1 and T_2 of U of size k such that their least common multiples do not divide each other then, $\mathbf{x}^{\mathbf{w}}$ is in image of $\phi_S^{(k)}$ on a syzygy between two minimal generators in $M_L^{(k)}$.*
3. *Otherwise, $\mathbf{x}^{\mathbf{w}}$ is in image of $\phi_S^{(k)}$ on a syzygy between a minimal generator in $M_L^{(k)}$ and $1_{\mathbb{K}}$.*

Proof. By definition, \mathbf{w} dominates at least $(k+1)$ points in L . Consider the $\binom{r}{k}$ subsets of U of size k and note that $\binom{r}{k} \geq 2$. For each subset T of size k , let ℓ_T be the least common multiple of the set of points in T . If the least common multiple ℓ_T is the same for all subsets T of size k , then we claim that $\mathbf{x}^{\mathbf{w}}$ is an exceptional generator of $M_L^{(k)}$. To see this, note that $\mathbf{x}^{\mathbf{w}} \in M_L^{(k)}$ and any minimal generator ℓ of $M_L^{(k)}$ that divides $\mathbf{x}^{\mathbf{w}}$ dominates every point in some subset of U of size k and ℓ is the least common multiple of the Laurent monomials corresponding to points in U . However, this least common multiple is $\mathbf{x}^{\mathbf{w}}$. Hence, $\ell = \mathbf{x}^{\mathbf{w}}$ and is an exceptional generator of $M_L^{(k)}$.

Otherwise, consider two subsets T_1 and T_2 of U of size k such that their least common multiples ℓ_{T_1} and ℓ_{T_2} respectively, are different. There are two cases:

Either ℓ_{T_1} and ℓ_{T_2} do not divide each other. Then both ℓ_{T_1} and ℓ_{T_2} are not equal to $\mathbf{x}^{\mathbf{w}}$ but divide it. Their supports (the set of lattice points that their exponents dominate) are precisely T_1 and T_2 respectively (otherwise, this would contradict $\mathbf{x}^{\mathbf{w}}$ being a minimal generator of $M_L^{(k+1)}$). Hence, ℓ_{T_1} and ℓ_{T_2} are minimal generators of $M_L^{(k)}$ as any Laurent

monomial that divides either ℓ_{T_1} or ℓ_{T_2} must have strictly smaller support. The map $\phi_S^{(k)}$ takes their syzygy to a monomial m that divides $\mathbf{x}^{\mathbf{w}}$. Furthermore, since this monomial m is in $M_L^{(k+1)}$ and $\mathbf{x}^{\mathbf{w}}$ is a minimal generator of $M_L^{(k+1)}$, we conclude that $m = \mathbf{x}^{\mathbf{w}}$. Finally, note that by Proposition 2.6 there is a lattice point $\mathbf{q} \in L$ such that $\ell_{T_1} \cdot \mathbf{x}^{-\mathbf{q}}$ and $\ell_{T_2} \cdot \mathbf{x}^{-\mathbf{q}}$ are minimal generators of $M_{L,\text{mod}}^{(k)}$. Their syzygy maps to an element in the same orbit as $\mathbf{x}^{\mathbf{w}}$ under the action of L .

Suppose that for every pair ℓ_{T_1} and ℓ_{T_2} one divides the other. Assume that ℓ_{T_1} is a proper divisor of ℓ_{T_2} and ℓ_{T_1} dominates exactly k points in L . Then ℓ_{T_2} along with the least common multiple of any other subset of size k other than T_1 is precisely $\mathbf{x}^{\mathbf{w}}$ (this is because for $\mathbf{x}^{\mathbf{w}}$ is a minimal generator for $M_L^{(k+1)}$). Hence, the least common multiple of the set of Laurent monomials with exponents in $T_1 \cup \{\mathbf{q}\}$ is $\mathbf{x}^{\mathbf{w}}$ for any $\mathbf{q} \in T_2 \setminus T_1$. The map $\phi_S^{(k)}$ takes the syzygy between the minimal generators $\ell_{T_1} \cdot \mathbf{x}^{-\mathbf{q}}$ and $1_{\mathbb{K}}$ of $M_{L,\text{mod}}^{(k)}$ to an element in the same orbit of $\mathbf{x}^{\mathbf{w}}$ under the action of the lattice L . □

Remark 2.8. Note that the proof of Theorem 2.7 also shows that any element in the image of $\phi_S^{(k)}$ satisfies Case 3 in Theorem 2.7 i.e., it is also in its image under a syzygy between a minimal generator of $M_L^{(k)}$ and $1_{\mathbb{K}}$. However, those that satisfy Case 2 also carry an L -action and hence, we have included this as a separate item in Theorem 2.7. □

Example 2.9. Consider the lattice $L = (3, 4, 11)^\perp \cap \mathbb{Z}^3$. Using our algorithm, we compute its 4th lattice module $M_L^{(4)}$ as an $S[L]$ -module equals $\langle x_3^2, x_1^{-1}x_2x_3^2, x_1^3x_2x_3 \rangle$. The minimal generator $x_1^{-1}x_2x_3^2$ dominates the lattice points $\{(-1, -2, 1), (-2, -4, 2), (-6, -1, 2), (-5, 1, 1)\}$ where there exists two 3-subsets whose least common multiples are distinct and proper divisors of $x_1^{-1}x_2x_3^2$. We observe that these subsets consist of the first three and last three lattice points, and give the following minimal generators of $M_{L,\text{mod}}^{(3)}$:

$$\begin{aligned} x_1^{-1}x_2^{-1}x_3^2 &= \text{lcm}(x_1^{-1}x_2^{-2}x_3, x_1^{-2}x_2^{-4}x_3^2, x_1^{-6}x_2^{-1}x_3^2) \\ x_1^{-2}x_2x_3^2 &= \text{lcm}(x_1^{-2}x_2^{-4}x_3^2, x_1^{-6}x_2^{-1}x_3^2, x_1^{-5}x_2x_3) \end{aligned}$$

Therefore $x_1^{-1}x_2x_3^2$ equals $\phi_S^{(3)}((x_1^{-1}x_2^{-1}x_3^2, x_1^{-2}x_2x_3^2))$ and so is realised as the image of a syzygy by two minimal generators of $M_L^{(3)}$, see Figure 1.

The minimal generator x_3^2 cannot be constructed in this way. It dominates the lattice points $\{(0, 0, 0), (-1, -2, 1), (-2, -4, 2), (-6, -1, 2)\}$ where only the least common multiple of the last three lattice points gives a proper divisor of x_3^2 , specifically $x_1^{-1}x_2^{-1}x_3^2$. This is a minimal generator of $M_{L,\text{mod}}^{(3)}$ and so x_3^2 equals $\phi_S^{(3)}((x_1^{-1}x_2^{-1}x_3^2, 1_{\mathbb{K}}))$, a syzygy between a minimal generator of $M_L^{(3)}$ and $1_{\mathbb{K}}$, as shown in Figure 2.

For an example of an exceptional generator, we look at the lattice $L = (2, 5, 10)^\perp \cap \mathbb{Z}^3$. The corresponding lattice ideal is $I_L = \langle x_3 - x_1^5, x_3 - x_2^2 \rangle$, therefore as an $S[L]$ -module $M_L^{(2)}$ has generators x_1^5, x_3, x_2^2 . These all lie in the same L -orbit and so $M_L^{(2)}$ is minimally generated by a single element x_3 . However x_3 dominates 3 lattice points $\{(0, 0, 0), (-5, 0, 1), (0, -2, 1)\}$. Therefore x_3 is an exceptional generator of $M_L^{(2)}$, as shown in Figure 3. Indeed, note that the

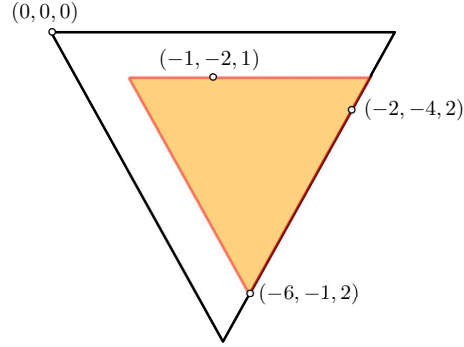


Figure 1: Minimal generator of $M_L^{(4)}$ realised as a syzygy between a minimal generator of $M_L^{(3)}$ and $1_{\mathbb{K}}$.

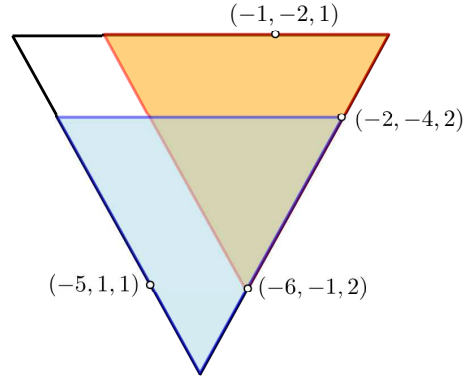


Figure 2: Minimal generator of $M_L^{(4)}$ realised as a syzygy between two minimal generators of $M_L^{(3)}$.

least common multiple of Laurent monomials corresponding to every pair of lattice points is also x_3 . \square

Note that $M_{L,\text{mod}}^{(k)}$ is not finitely generated as an S -module and is also not an $S[L]$ -module. This makes Theorem 2.7 somewhat unwieldy to compute $M_L^{(k)}$. In the following, we use Theorem 2.7 to prove the neighbourhood theorem that is computationally more amenable.

2.3 Neighbourhood Theorem

We briefly recall the graph G_L induced on the lattice L . Fix a binomial minimal generating set B of I_L . There is an edge between points \mathbf{w}_1 and \mathbf{w}_2 in L if there exists a minimal generator $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in B$ such that $\mathbf{u} - \mathbf{v} = \mathbf{w}_1 - \mathbf{w}_2$. Let d_{G_L} be the metric on L induced by the graph G_L . For a point $\mathbf{w} \in L$, we define $N^{(k)}(\mathbf{w})$ to be the set of all points in L in the ball of radius k with respect to the metric d_{G_L} and with \mathbf{w} as its center.

Theorem 2.10. (Neighbourhood Theorem) *Any minimal generator of $M_L^{(k)}$ as an $S[L]$ -module is the least common multiple of Laurent monomials corresponding to k lattice points*

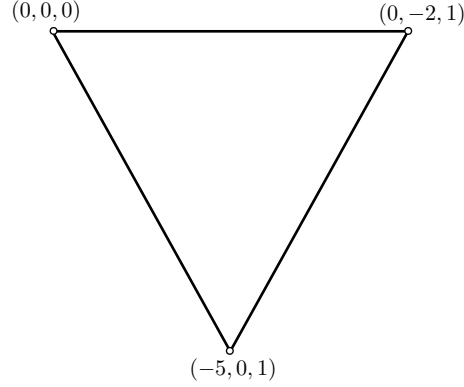


Figure 3: Exceptional generator of $M_{L(2,5,10)}^{(2)}$.

in $N^{(k-1)}(\mathbf{0})$. Equivalently, any minimal generator of $M_L^{(k)}$ as an S -module is the least common multiple of Laurent monomials corresponding to k lattice points in $N^{(k-1)}(\mathbf{q})$.

In order to prove the theorem, we study certain “local pieces” of G_L called the fiber graph.

Definition 2.11. [18, Page 39] Let $A = (a_1, \dots, a_n)$. For each non-negative integer b we define the set $\mathcal{F}_b = \{\mathbf{u} \in \mathbb{Z}_{\geq 0}^n : A \cdot \mathbf{u} = b\}$ to be the fiber of A over b .

For any lattice point $\mathbf{u} \in L$, we can express it uniquely as the difference of positive and negative parts $\mathbf{u}^+ - \mathbf{u}^-$, where the i -th coordinate of \mathbf{u}^+ equals u_i if $u_i > 0$ and equals 0 otherwise. Since L is contained in $(a_1, \dots, a_n)^\perp$, we have $\mathbf{u}^+ \in \mathcal{F}_b$ if and only if $\mathbf{u}^- \in \mathcal{F}_b$.

We induce a natural graph on the fiber, denoted the fiber graph G_b . Fix a binomial minimal generating set B of I_L . The vertices of the graph are the elements of the fiber \mathcal{F}_b with an edge between \mathbf{w}_1 and \mathbf{w}_2 if there exists a minimal generator $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in B$ such that $\mathbf{u} - \mathbf{v} = \mathbf{w}_1 - \mathbf{w}_2$. We note that G_b is a finite graph that can be embedded into G_L . The following lemma generalizes the statement [18, Theorem 5.3] that if I_L is a prime ideal (equivalently, if L is a saturated lattice) then \mathcal{F}_b is connected.

Lemma 2.12. Let $\mathbf{u}, \mathbf{v} \in \mathcal{F}_b$. The difference $\mathbf{u} - \mathbf{v}$ is a lattice point (a point in L) if and only if \mathbf{u}, \mathbf{v} are in the same connected component of G_b .

Proof. Suppose $\mathbf{u} - \mathbf{v} \in L$, then by definition $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_L$ and so can be represented as an S -linear combination of the minimal generators:

$$\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = \sum_{i=1}^N \mathbf{x}^{\mathbf{w}_i} \cdot (\mathbf{x}^{\mathbf{g}_i^+} - \mathbf{x}^{\mathbf{g}_i^-}) \quad (1)$$

We will show by induction on N there exists a path in G_b between \mathbf{u} and \mathbf{v} . For $N = 1$, expression (1) is equivalent to saying that $\mathbf{u} - \mathbf{v} = \mathbf{g}_i$ and so they must be connected by an edge.

Assume the induction hypothesis holds for all $N < N'$, consider expression (1) for $N = N'$. We have $\mathbf{x}^{\mathbf{u}} = \mathbf{x}^{\mathbf{w}_1} \cdot \mathbf{x}^{\mathbf{g}_1^+}$ for some i , so without loss of generality we say that $\mathbf{u} = \mathbf{w}_1 + \mathbf{g}_1^+$, implying \mathbf{u} and $\mathbf{w}_1 + \mathbf{g}_1^-$ are connected by an edge. Subtracting $\mathbf{x}^{\mathbf{w}_1} \cdot (\mathbf{x}^{\mathbf{g}_1^+} - \mathbf{x}^{\mathbf{g}_1^-})$ from (1) gives us an expression of length $N' - 1$ for $\mathbf{x}^{\mathbf{w}_1 + \mathbf{g}_1^-} - \mathbf{x}^{\mathbf{v}}$. By the induction hypothesis, these exponents are connected and so \mathbf{u} and \mathbf{v} must also be connected.

Conversely, assume that \mathbf{u}, \mathbf{v} are in the same connected component of G_b . Then there exists some path $\mathbf{u} = v^{(0)}, v^{(1)}, \dots, v^{(N)} = \mathbf{v}$ in G_b . We can write the binomial

$$\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = \sum_{i=1}^N \mathbf{x}^{v^{(i-1)}} - \mathbf{x}^{v^{(i)}}$$

where each binomial $\mathbf{x}^{v^{(i-1)}} - \mathbf{x}^{v^{(i)}}$ is an element of I_L , as $v^{(i-1)}, v^{(i)}$ are connected by an edge. Therefore, $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_L$ and so $\mathbf{u} - \mathbf{v} \in L$. \square

Lemma 2.13. *Let \mathbf{v} be a lattice point with $\mathbf{v}^+, \mathbf{v}^- \in F \subseteq \mathcal{F}_b$, where F is a subset of the fiber \mathcal{F}_b consisting of all elements in the same connected component of G_b . The exponent of the least common multiple $\text{lcm}(\mathbf{x}^{\mathbf{v}}, 1_{\mathbb{K}})$ dominates precisely $|F|$ lattice points, specifically those of the form $\mathbf{v}^+ - \mathbf{u}$ where $\mathbf{u} \in F$.*

Proof. We first observe that $\text{lcm}(\mathbf{x}^{\mathbf{v}}, 1_{\mathbb{K}}) = \mathbf{x}^{\mathbf{v}^+}$. Let $\mathbf{u} \in F$, then by Lemma 2.12 we deduce that $\mathbf{v}^+ - \mathbf{u} \in L$. As $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$, we see $\mathbf{v}^+ \geq \mathbf{v}^+ - \mathbf{u}$. This holds for every $\mathbf{u} \in F$ and so the exponent of $\text{lcm}(\mathbf{x}^{\mathbf{v}}, 1_{\mathbb{K}})$ dominates at least $|F|$ lattice points. Conversely, suppose that for some $\mathbf{p} \in L$, $\mathbf{v}^+ \geq \mathbf{p}$. Let $\mathbf{u} = \mathbf{v}^+ - \mathbf{p} \in \mathbb{Z}_{\geq 0}^n$. Then $\mathbf{v}^+ - \mathbf{u} \in L$, hence by Lemma 2.12 $\mathbf{u} \in F$. \square

Lemma 2.14. *Let $\mathbf{u}, \mathbf{v} \in L$, $d_{G_L}(\mathbf{u}, \mathbf{v}) = k$. There exists a path of length at least k in G_L from \mathbf{u} to \mathbf{v} such that the exponent of $\text{lcm}(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}})$ dominates every lattice point on the path.*

Proof. As G_L is invariant under translation by L , it suffices to prove the case where $\mathbf{u} = \mathbf{0}$. Assume $\mathbf{v}^+, \mathbf{v}^- \in \mathcal{F}_b$, by Lemma 2.12 they lie in the same connected component of G_b and so there exists a path in G_b given by $\mathbf{v}^+ = v^{(0)}, v^{(1)}, \dots, v^{(n)} = \mathbf{v}^-$. We can embed this path into G_L by the embedding $\mathbf{v}^+ - v^{(i)}$. This gives us a path from $\mathbf{0}$ to \mathbf{v} in G_L and by Lemma 2.13 the exponent of $\text{lcm}(1_{\mathbb{K}}, \mathbf{x}^{\mathbf{v}})$ dominates each of the lattice points on this path. As $d_{G_L}(\mathbf{0}, \mathbf{v}) = k$, this path must be at least length k . \square

Proof. (Proof of Theorem 2.10) We proceed by induction on k . For the base case of $k = 1$, the lattice module $M_L^{(1)} = M_L$ has a single generator $1_{\mathbb{K}}$ corresponding to the single lattice point in $N^{(0)}(\mathbf{0})$. Assume the statement is true for all $k \leq k_0$. Let $\mathbf{x}^{\mathbf{u}}$ be a minimal generator of $M_L^{(k_0+1)}$, then by Theorem 2.7 this is either in the image of the map $\phi_S^{(k_0)}$ or is an exceptional generator of $M_L^{(k_0)}$.

Suppose that it is an exceptional generator of $M_L^{(k_0)}$, then by the inductive hypothesis $\mathbf{x}^{\mathbf{u}}$ can be expressed as the least common multiple of Laurent monomials corresponding to a set of precisely k_0 lattice points, which we denote as $P_{\mathbf{u}}$. Note that $P_{\mathbf{u}}$ is a proper subset of the support of $\mathbf{x}^{\mathbf{u}}$. By lattice translation, we assume that $P_{\mathbf{u}}$ is contained in $N^{(k_0-1)}(\mathbf{0})$ and contains $\mathbf{0}$. It suffices to show that \mathbf{u} dominates another lattice point in $N^{(k_0)}(\mathbf{0})$.

As an exceptional generator $\mathbf{x}^{\mathbf{u}}$ must dominate at least $k_0 + 1$ lattice points, so consider a lattice point $\mathbf{p} \notin P_{\mathbf{u}}$ that is dominated by \mathbf{u} . If $\mathbf{p} \in N^{(k_0)}(\mathbf{0})$, we are done. Suppose $\mathbf{p} \in N^{(r)}(\mathbf{0})$, $r > k_0$. By Lemma 2.14 there exists a path from \mathbf{p} to $\mathbf{0}$ in G_L such that every lattice point in the path is dominated by the exponent of $\text{lcm}(\mathbf{x}^{\mathbf{p}}, 1_{\mathbb{K}})$. Therefore there exists some lattice point \mathbf{q} in this path with $d_{G_L}(\mathbf{q}, \mathbf{0}) = k_0$ that is dominated by the exponent of $\text{lcm}(\mathbf{x}^{\mathbf{p}}, 1_{\mathbb{K}})$. Furthermore as $P_{\mathbf{u}}$ is contained in $N^{(k_0-1)}(\mathbf{0})$, $\mathbf{q} \notin P_{\mathbf{u}}$. As $\text{lcm}(\mathbf{x}^{\mathbf{p}}, 1_{\mathbb{K}})$ divides $\mathbf{x}^{\mathbf{u}}$, it must also dominate all lattice points along this path. Therefore $\mathbf{x}^{\mathbf{u}}$ can be written as the least common multiple of the Laurent monomials corresponding to the lattice points $P_{\mathbf{u}} \cup \{\mathbf{q}\}$ whose cardinality is $k_0 + 1$.

Suppose $\mathbf{x}^{\mathbf{u}}$ is in the image of $\phi_S^{k_0}$. According to Remark 2.8, $\mathbf{x}^{\mathbf{u}}$ is the image of a syzygy between one minimal generator of $M_L^{(k_0)}$ as an S -module and $1_{\mathbb{K}}$. This minimal generator is in the same L -orbit as $\mathbf{x}^{\mathbf{v}}$, a minimal generator of $M_L^{(k_0)}$ satisfying the induction hypothesis. More precisely, there exists a set $P_{\mathbf{v}}$ of k_0 lattice points whose least common multiple of Laurent monomials equals $\mathbf{x}^{\mathbf{v}}$ is contained in $N^{(k_0-1)}(\mathbf{0})$ and contains $\mathbf{0}$. Hence, $\mathbf{x}^{\mathbf{u}}$ is in the same L -orbit as $\text{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{p}})$ for some lattice point \mathbf{p} . It suffices to show that $\text{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{p}})$ satisfies the statement of the theorem.

Let $\mathbf{p} \in N^{(r)}(\mathbf{0})$, if $r \leq k_0$ then we are done. Suppose $r > k_0$, by Lemma 2.14 there exists a path from $\mathbf{0}$ to \mathbf{p} in G_L such that every lattice point in the path is dominated by the exponent of $\text{lcm}(1_{\mathbb{K}}, \mathbf{x}^{\mathbf{p}})$. By the same argument as the previous case, there exists a lattice point \mathbf{q} on this path with $d_{G_L}(\mathbf{0}, \mathbf{p}) = k_0$, that is necessarily dominated by \mathbf{u} and not contained in $P_{\mathbf{v}}$. Therefore $\text{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{q}})$ is the least common multiple of the Laurent monomials corresponding to $k_0 + 1$ lattice points $P_{\mathbf{v}} \cup \{\mathbf{q}\}$. The monomial $\text{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{q}})$ divides $\text{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{p}})$, and so is equal by minimality of $\text{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{p}})$. Therefore $\text{lcm}(\mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{p}})$ is the least common multiple of $k_0 + 1$ Laurent monomials corresponding to $P_{\mathbf{v}} \cup \{\mathbf{q}\}$ contained in $N^{(k_0)}(\mathbf{0})$. □

From the proof of the neighbourhood theorem, we see that the following slightly stronger statement also holds: every minimal generator of $M_L^{(k)}$ as an $S[L]$ -module is the least common multiple of Laurent monomials corresponding to k -points in $N^{(k-1)}(\mathbf{0})$ one of which is the origin.

3 Finiteness Results

In this section, we show that after suitable twists there are only finitely many isomorphism classes of generalised lattice modules. More precisely, we show the following:

Theorem 3.1. *Let L be a lattice of the form $(a_1, \dots, a_n)^{\perp} \cap \mathbb{Z}^n$. For each $k \in \mathbb{N}$, let $\mathbf{x}^{\mathbf{u}_k}$ be any element $M_L^{(k)}$ of the smallest (a_1, \dots, a_n) -weighted degree. There are finitely many classes among the generalised lattice modules $\{M_L^{(k)}(-\mathbf{u}_k)\}_{k \in \mathbb{N}}$ up to isomorphism of both \mathbb{Z}^n -graded $S[L]$ -modules and \mathbb{Z}^n -graded S -modules.*

The main ingredient of the proof of Theorem 3.1 is the structure poset of L that we briefly recall.

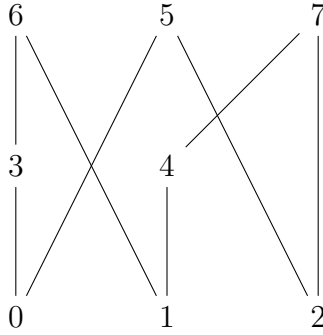


Figure 4: The structure poset of $L(3, 5, 8)$.

3.1 Structure Poset of L

The elements of the structure poset of L are elements in \mathbb{Z}^n/L of (a_1, \dots, a_n) -weighted degree in the range $[0, F_1]$ where F_1 is the first Frobenius number of L . The partial order in this poset is defined as follows: for elements $[\mathbf{a}]$, $[\mathbf{b}]$ in the structure poset we say that $[\mathbf{a}] \geq [\mathbf{b}]$ if for every representative $\mathbf{a} \in \mathbb{Z}^n$ of $[\mathbf{a}]$ there exists a representative \mathbf{b} of $[\mathbf{b}]$ such that $\mathbf{a} \geq \mathbf{b}$. Note, $[\mathbf{a}] \geq [\mathbf{b}]$ if and only if $[\mathbf{a} - \mathbf{b}] \geq [\mathbf{0}]$. Hence, the structure poset of L can be constructed from the set of all elements $[\mathbf{a}] \geq [\mathbf{0}]$ in \mathbb{Z}^n/L whose (a_1, \dots, a_n) -weighted degree is in the range $[0, F_1]$. This observation is useful to compute the structure poset.

Example 3.2. Let $(a_1, a_2, a_3) = (3, 5, 8)$ and hence, $L(3, 5, 8) = (3, 5, 8)^\perp \cap \mathbb{Z}^3$. The first Frobenius number is 7. Hence, the structure poset of L consists of eight elements labelled 0 to 7. The poset relations can be determined from the set of all elements that dominate 0, in this case they are 3, 5, 6. The Haase diagram of the structure poset is shown in Figure 3.1. \square

Recall that m_k is the minimum (a_1, \dots, a_n) -weighted degree of any element of $M_L^{(k)}$. This is the smallest value j such that the Hilbert coefficient h_j of the polynomial ring S with (a_1, \dots, a_n) -weighted degree is at least k (this Hilbert series is also referred as the restricted partition function in [8, Page 6]). The key observation is that $M_L^{(k)}$ is completely determined (up to isomorphism of \mathbb{Z}^n -graded $S[L]$ -modules) by “filling” the structure poset of L . More precisely, $M_L^{(k)}$ is determined by the elements in \mathbb{Z}^n/L of weighted degree $[m_k, m_k + F_1]$ that dominate at least k -points in L . This information is also given by the Hilbert function of the polynomial ring with (a_1, \dots, a_n) -weighted degree. This determines a poset by exactly the same partial order as the structure poset of L . Furthermore, by taking $m_k + i$ to i , this determines a subposet of the structure poset of L that we refer to as the *structure poset of $M_L^{(k)}$* . In particular, the minimal generators of $M_L^{(k)}$ correspond to the minimal elements of its structure poset.

Proof. (Proof of Theorem 3.1) Note that for any k , the (a_1, \dots, a_n) -weighted degree of the minimal generators of $M_L^{(k)}$ are in the range $[m_k, m_k + F_1]$. Furthermore, the structure poset

of $M_L^{(k)}$ as a subposet of the structure poset of L determines $M_L^{(k)}$ up to isomorphism of \mathbb{Z}^n -graded $S[L]$ -modules. More precisely, if $M_L^{(k_1)}$ and $M_L^{(k_2)}$ have the same structure poset, then multiplying $M_L^{(k_1)}(-\mathbf{u}_1)$ by the Laurent monomial $\mathbf{x}^{\mathbf{u}_2}/\mathbf{x}^{\mathbf{u}_1}$ is an isomorphism between $M_L^{(k_1)}(-\mathbf{u}_1)$ and $M_L^{(k_2)}(-\mathbf{u}_2)$ (as both \mathbb{Z}^n -graded $S[L]$ -modules and \mathbb{Z}^n -graded S -modules). In particular, this map induces a bijection between the (monomial) minimal generating of $M_L^{(k_1)}(-\mathbf{u}_1)$ and the (monomial) minimal generating set of $M_L^{(k_2)}(-\mathbf{u}_2)$ and preserves degrees. Since the structure poset of L is finite, it has only finitely many subposets. Hence, there are only finitely many \mathbb{Z}^n -graded isomorphism classes of the twisted generalised lattice modules $\{M_L^{(k)}(-\mathbf{u}_k)\}_{k=1}^\infty$. \square

Theorem 3.1 and its proof also generalises to finite index sublattices L of $(a_1, \dots, a_n)^\perp \cap \mathbb{Z}^n$. The only additional subtlety is that the structure poset of $M_L^{(k)}$ will have precisely as many embeddings into the structure poset of L as the number of elements of weighted degree m_k in $M_L^{(k)}$. If $M_L^{(k_1)}$ and $M_L^{(k_2)}$ have the same embedding into the structure poset of L , then we have exactly the same isomorphism as in the proof of Theorem 3.1. There are still only finitely many subposets of the structure poset of L .

Example 3.3. The number of subposets of a poset is equal to the number of antichains. For counting the number of antichains, tools such as Dilworth's theorem [9] are useful. In the following, we compute the structure poset of $M_{L(3,5,8)}^{(k)}$ for k from 1 to 6. The key input is that the Hilbert series of polynomial ring with the (a_1, \dots, a_n) -weighted grading is given by the rational function $\frac{1}{(1-t)^{a_1} \dots (1-t)^{a_n}}$. Using this information, we determine m_1, \dots, m_6 to be 0, 8, 16, 21, 24, 29. The corresponding structure posets are shown in Figure 3.1. \square

Based on the same ideas as in Theorem 3.1, we obtain the following upper bounds on generalised Frobenius numbers and the number of minimal generators of $M_L^{(k)}$.

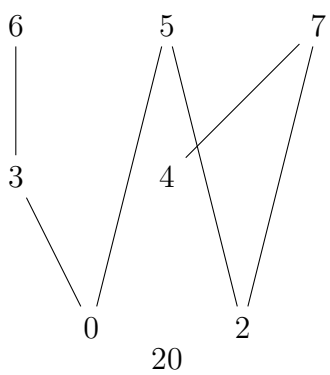
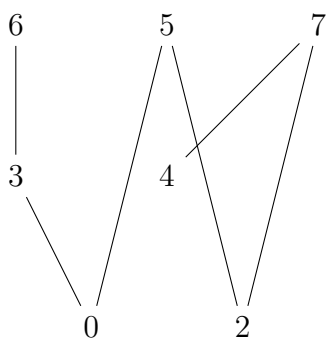
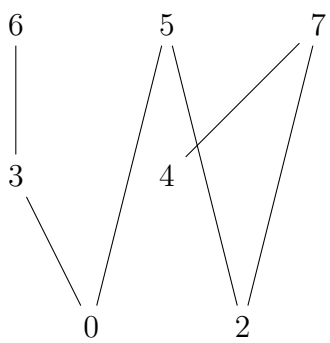
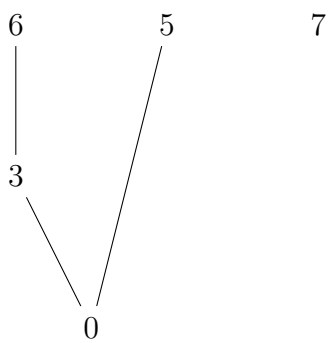
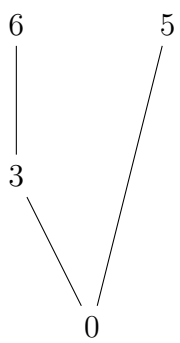
Proposition 3.4. *The k -th Frobenius number F_k is upper bounded by $m_k + F_1$. The number $\beta_1(M_L^{(k)})$ of minimal generators of $M_L^{(k)}$ as an $S[L]$ -module is upper bounded by the maximum length of an antichain in the structure poset of L .*

Furthermore, we have the following corollary to Theorem 3.1.

Corollary 3.5. *There exists a finite set of integers $\{b_1, \dots, b_t\} \subset \mathbb{Z}_{\geq 0} \cup \{-1\}$ such that for every k there exists a natural number j such that the k -th Frobenius number can be written as:*

$$F_k = m_k + b_j$$

where m_k is the minimum (a_1, \dots, a_n) -weighted degree of an element in $M_L^{(k)}$. This finite set $\{b_1, \dots, b_t\}$ is precisely the set of natural numbers that can be realised as $\text{reg}(M_L^{(k)}(-\mathbf{u}_k)) - n - 1 + \sum_{i=1}^n a_i$.



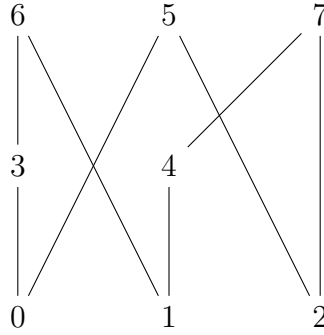


Figure 5: The structure posets of $M_{L(3,5,8)}^{(k)}$ for k from 1 to 6 (with k increasing from top to bottom).

4 Applications

4.1 The Sequence of Generalised Frobenius Numbers

We prove that the sequence of generalised Frobenius numbers form a generalised arithmetic progression.

Theorem 4.1. *For any finite index sublattice L of $(a_1, \dots, a_n)^\perp \cap \mathbb{Z}^n$, the sequence of generalised Frobenius numbers $(F_k)_{k=1}^\infty$ is a generalised arithmetic progression.*

We note that this follows immediately from Corollary 3.5, once we show that the sequence $(m_k)_{k=1}^\infty$ is also a generalised arithmetic progression.

Lemma 4.2. *For any finite index sublattice L of $(a_1, \dots, a_n)^\perp \cap \mathbb{Z}^n$, the sequence $(m_k)_{k=1}^\infty$ is a generalised arithmetic progression. In particular, for all $k \in \mathbb{N}$ we have $m_2 \geq m_{k+1} - m_k \geq 0$.*

Proof. The inequality $m_{k+1} - m_k \geq 0$ follows by construction. In the following, we show that $m_{k+1} - m_k \leq m_2$. Consider a minimal generator of $M_L^{(2)}$ of weighted degree m_2 that dominates the origin and another lattice point \mathbf{p} . Note that this minimal generator is $\mathbf{x}^{\mathbf{p}^+}$. Consider a minimal generator $\mathbf{x}^{\mathbf{q}}$ of $M_L^{(k)}$ of weighted degree m_k , such that the origin is in its support and \mathbf{p} is not in its support. Note that such a generator exists by a lattice translation argument. More precisely, take any minimal generator of $M_L^{(k)}$ and maximize the linear functional $\mathbf{p} \cdot \mathbf{x}$ over its support. Suppose that \mathbf{r} is such a point (in L), multiply the minimal generator by $\mathbf{x}^{-\mathbf{r}}$. The resulting minimal generator contains the origin but does not contain the point \mathbf{p} in its support. This is because the inner product of \mathbf{p} with the origin is zero whereas its inner product with itself is strictly positive. The polynomial $\text{lcm}(\mathbf{x}^{\mathbf{p}^+}, \mathbf{x}^{\mathbf{q}})$ is contained in $M_L^{(k+1)}$ and has weighted degree at most $m_2 + m_k$. \square

The sequence $(m_k)_{k=1}^\infty$ inherits much of the structure of $M_L^{(k)}$ given by its inductive characterisation (Theorem 2.7). This additional structure makes it more natural to derive bounds on successive differences rather than $(F_k)_{k=1}^\infty$ directly.

The dimension of the generalised arithmetic progression is defined as the cardinality of its set of successive differences. Note that the dimension is equal to one when the sequence is an arithmetic progression. Given the sequence of k -th Frobenius numbers $(F_k)_{k=1}^\infty$ with associated $\{b_1, \dots, b_t\}$ such that $b_t \geq b_{t-1} \geq \dots \geq b_1$, we derive two upper bounds on its dimension from Lemma 4.2 and Corollary 3.5.

$$\begin{aligned}\dim((F_k)_{k=1}^\infty) &\leq t(m_2 + 1) \\ \dim((F_k)_{k=1}^\infty) &\leq m_2 + b_t - b_1 + 1\end{aligned}$$

Corollary 4.3. *A geometric progression with common ratio strictly greater than one cannot occur as a sequence of generalised Frobenius numbers of any finite index sublattice of $(a_1, \dots, a_n)^\perp \cap \mathbb{Z}^n$.*

Proof. By Theorem 4.1, a sequence of generalised Frobenius numbers $(F_k)_{k=1}^\infty$ is a generalised arithmetic progression. Hence, the difference $F_{k+1} - F_k$ is uniformly upper bounded. On the other hand, since the common ratio of the geometric progression is greater than one, the difference between successive terms goes to infinity with k . Hence, such a geometric progression cannot occur as a sequence of generalised Frobenius numbers. \square

Remark 4.4. Another reason to expect Corollary 4.3 is that the sequence of generalised Frobenius numbers of lattices of dimension at least two usually contains plenty of repetitions. However, Theorem 4.1 implies a stronger statement that even after removing the repetitions the resulting sequence cannot be a geometric progression of common ratio strictly greater than one. \square

4.2 Algorithms for Generalised Frobenius Numbers

We use the Neighbourhood theorem (Theorem 2.10) to give an algorithmic construction of generalised lattice modules and via Proposition 1.6 compute generalised Frobenius numbers.

Remark 4.5. A method for computing the lattice ideal given a basis for that lattice is presented in [13]. One method to compute the Castelnuovo-Mumford regularity of $\pi(M_L^{(k)})$ is to construct a free presentation of $\pi(M_L^{(k)})$, for instance via the hull complex of $M_L^{(k)}$. We can use this as the input to the algorithm presented in [5] to compute the Castelnuovo-Mumford regularity. \square

Example 4.6. In the following example, we illustrate our algorithm in the case where the lattice $L = L(3, 4, 11) = (3, 4, 11)^\perp \cap \mathbb{Z}^3$ and $k = 3$. The set $\{(1, 2, -1), (4, -3, 0)\}$ is a basis for L .

The binomials corresponding to this basis generate the ideal $J = \langle x_1x_2^2 - x_3, x_1^4 - x_2^3 \rangle$. The lattice ideal I_L is given by the saturation of J with respect to the product of all the variables, and so

$$I_L = \langle J : \langle x_1x_2x_3 \rangle^\infty \rangle = \langle x_1x_2^2 - x_3, x_1^4 - x_2^3 \rangle.$$

In this case, the lattice ideal does not have any new binomials.

Algorithm 1 Generalised Lattice Modules

- 1: **Input:** A basis of a finite index sublattice L of $(a_1, \dots, a_n)^\perp \cap \mathbb{Z}^n$ where $(a_1, \dots, a_n) \in \mathbb{N}^n$ and a natural number $k \in \mathbb{N}$.
 - 2: **Output:** A minimal generating set of $M_L^{(k)}$ as an $S[L]$ -module and the k -th Frobenius number F_k of L .
 - 3: Compute the lattice ideal I_L .
 - 4: Compute all lattice points in $N^{(k-1)}(\mathbf{0})$.
 - 5: For each k -subset $P \subseteq N^{(k-1)}(\mathbf{0})$ containing $\mathbf{0}$, calculate the least common multiple $\ell_P = \text{lcm}(\mathbf{x}^{\mathbf{p}_i} \mid \mathbf{p}_i \in P)$.
 - 6: Construct $M_L^{(k)} = \langle \ell_P \mid P \subseteq N^{(k-1)}(\mathbf{0}), |P| = k, \mathbf{0} \in P \rangle_{S[L]}$.
 - 7: Pick a representative for each L -orbit and declare the resulting set to be a minimal generating set of $M_L^{(k)}$.
 - 8: Compute the \mathbb{Z}^n/L -graded S -module $\pi(M_L^{(k)}) := M_L^{(k)} \otimes_{S[L]} S$ and its Castelnuovo-Mumford regularity $\text{reg}(\pi(M_L^{(k)}))$.
 - 9: Set the k -th Frobenius number F_k to $\text{reg}(\pi(M_L^{(k)})) + n - 1 - \sum_{i=1}^n a_i$.
-

The lattice points $(1, 2, -1), (4, -3, 0)$ along with their negative and the origin $\mathbf{0}$, give the first neighbourhood $N^{(1)}(\mathbf{0})$. Next, we compute $N^{(k-1)}(\mathbf{0})$ by taking all k -subsets of $N^{(1)}(\mathbf{0})$ and taking their sum. This computation gives us

$$N^{(2)}(\mathbf{0}) = \{(0, 0, 0), (1, 2, -1), (4, -3, 0), (-1, -2, 1), (-4, 3, 0), (8, -6, 0), (3, -5, 1), (5, -1, -1), (-2, -4, 2), (2, 4, -2), (-5, 1, 1), (-3, 5, -1), (-8, 6, 0)\}.$$

For each 3-subset of $N^{(2)}(\mathbf{0})$, we take the least common multiple of the corresponding monomials and denote the $S[L]$ -module generated by these monomials as M_{con} . By the Neighbourhood theorem, M_{con} is equal to $M_L^{(3)}$. Note that this requires computing $\binom{12}{2} = 66$ monomials.

To calculate a minimal generating set of $M_L^{(3)}$, we choose the monomials from this set that do not dominate any other monomial in $M_L^{(3)}$. In our case, this gives the following list of generators

$$M_L^{(3)} = \langle x_1^5, x_1^4 x_2^2, x_1 x_2^3, x_1^3 x_3, x_2^5, x_2 x_3 \rangle_{S[L]}.$$

All minimal generators with the same \mathbb{Z}^n/L -degree must be in the same L -orbit. Hence, we pick representatives for each degree to give a minimal generating set of $M_L^{(3)}$. All minimal generators are in degree 15 or 20, and so $M_L^{(3)} = \langle x_1^5, x_1^4 x_2^2 \rangle_{S[L]}$. We compute the Castelnuovo-Mumford regularity of $(\pi(M_L^{(3)})) = 33$. Therefore, we calculate F_3 to be

$$33 + 2 - 3 - 4 - 11 = 17$$

□

5 Future Directions

We organise potential future directions into three items with the first two closely related.

- **Classification of Sequences of Generalised Frobenius Numbers:** We have shown that the sequence of generalised Frobenius numbers form a generalised arithmetic progression, however there is still information that we have not fully utilised. For instance, we have not used the filtration of the generalised lattice modules and the inductive characterisation provided by Theorem 2.10. Can this information be used to study sequences of generalised Frobenius numbers? For instance, by studying the sequence of Castelnuovo-Mumford regularity of modules in a filtration.
- **Syzygies of Generalised Lattice Modules:** Our finiteness result shows that for any finite index sublattice of $(a_1, \dots, a_n)^\perp \cap \mathbb{Z}^n$ there are only finitely many isomorphism classes of generalised lattice modules. What are the possible Betti tables that can occur as Betti tables of generalised lattice modules? How are they related? Note that this is closely related to the previous item since the Castelnuovo-Mumford regularity of $M_L^{(k)}$ is the number of rows of its Betti table minus one and this is essentially the k -th Frobenius number (Proposition 1.6). This problem is also closely related to the problem of classifying structure posets of generalised lattice modules (see Subsection 3.1 for more details).

Peleva and Sturmfels [14] define a notion of lattice ideals associated to generic lattices and show that the Scarf complex minimally resolves lattice ideals associated to generic lattices. For any fixed k and a generic lattice L , is there a generalisation of the Scarf complex to a complex that minimally resolves $M_L^{(k)}$ as an $S[L]$ -module?

- **Generalised Frobenius Numbers of Laplacian Lattices:** Let G be a labelled graph. Recall that the Laplacian matrix $Q(G)$ is the matrix $D - A$ where D is the diagonal matrix $\text{diag}(\text{val}(v_1), \dots, \text{val}(v_n))$ where $\text{val}(v_i)$ is the valency of the vertex v_i and A is the vertex-vertex adjacency matrix. The Laplacian lattice L_G of G is the lattice generated by the rows of the Laplacian matrix. This is a finite index sublattice of the root lattice $A_{n-1} = (1, \dots, 1)^\perp \cap \mathbb{Z}^n$ of index equal to the number of spanning trees of G . We know from [1] that the first Frobenius number of L_G is equal to the genus of the graph. The genus of the graph is its first Betti number as a simplicial complex of dimension one and is equal to $m - n + 1$ where m is the number of edges. Is there a generalisation of this interpretation to generalised Frobenius numbers?

Arithmetical graphs are generalisations of graphs motivated by applications from arithmetic geometry, see Lorenzini [11]. Lorenzini associated a Laplacian lattice to an arithmetical graph and defines its genus as the first Frobenius number of its Laplacian lattice. He studies it in the context of the Riemann-Roch theorem. The generalised Frobenius numbers of Laplacian lattices associated to arithmetical graphs seems another fruitful future direction.

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